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Coset spaces and representatives: a simple example

Let us consider the case of the SL(2)/SO(2) coset space. Even though its simplicity, it already shows the consequences of the choice of parameterisation. In the Cartan basis, the algebra of SL(2) is spanned by the Cartan generator H together with the positive E_+ and the negative E_- roots

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \, , \tag{1}$$

with non-vanishing commutation relations of the form $[H, E_\pm] = \pm 2\, E_\pm$ and $[E_+, E_-] = H$.

The standard *geometric* choice

The standard choice of coset representative is obtained by exponentiating the Cartan and the positive root generators in (1)

$$\mathcal{V}_g = e^{\phi_g/2\, H}\, e^{\chi_g\, E_+} = \begin{pmatrix} e^{\phi_g/2} & e^{\phi_g/2}\, \chi_g \\ 0 & e^{-\phi_g/2} \end{pmatrix} , \tag{2}$$

so that global SL(2) transformations Λ act on it from the right and compensating (local) SO(2) transformations O from the left, *i.e.* $\mathcal{V}_g \to \mathcal{V}_g' = O \mathcal{V}_g \Lambda$. In a more geometrical interpretation, $\mathcal{V}_g$ encodes the real fractional transformation $S = (dS_0 + b)/(cS_0 + a)$, with $ad - bc = 1$, needed to connect the choice of identity element $S_0 = i$ to any other point $S = \chi_g + i e^{-\phi_g}$ in the upper-half complex plane.

Using this representative, one can construct the scalar matrix

$$\mathcal{M}_g = \mathcal{V}_g^t\, \mathcal{V}_g = \begin{pmatrix} e^{\phi_g} & e^{\phi_g}\, \chi_g \\ \chi_g\, e^{\phi_g} & e^{-\phi_g} + \chi_g\, e^{\phi_g}\, \chi_g \end{pmatrix} \, , \tag{3}$$

upon which SL(2) transformations have a linear action $\mathcal{M}_g \to \Lambda^t \mathcal{M}_g \Lambda$, and define the SL(2) invariant non-linear sigma model

$$L_g = \frac{1}{8}\, \mathrm{Tr}\, \partial \mathcal{M}_g \partial \mathcal{M}_g^{-1} = -\frac{1}{4} \left[(\partial \phi_g)^2 + e^{2\phi_g} (\partial \chi_g)^2 \right] \, . \tag{4}$$

However, any choice of coset representative is as good as any other: they will be connected via a local (field dependent) SO(2) transformation combined with a field redefinition.

The standard *non-geometric* choice

Let us build the representative $\mathcal{V}_{ng}$ by exponentiating the Cartan and the negative root generators in (1)

$$\mathcal{V}_{ng} = e^{\phi_{ng}/2\,H}\, e^{\chi_{ng}\,E_-} = \begin{pmatrix} e^{\phi_{ng}/2} & 0 \\ e^{-\phi_{ng}/2}\,\chi_{ng} & e^{-\phi_{ng}/2} \end{pmatrix} . \tag{5}$$

The scalar matrix is now given by

$$\mathcal{M}_{ng} = \mathcal{V}_{ng}^t\,\mathcal{V}_{ng} = \begin{pmatrix} e^{\phi_{ng}} + \chi_{ng}\,e^{-\phi_{ng}}\,\chi_{ng} & \chi_{ng}\,e^{-\phi_{ng}} \\ e^{-\phi_{ng}}\,\chi_{ng} & e^{-\phi_{ng}} \end{pmatrix} , \tag{6}$$

and the SL(2) invariant non-linear sigma model reads

$$L_{ng} = \frac{1}{8}\,\mathrm{Tr}\,\partial\mathcal{M}_{ng}\partial\mathcal{M}_{ng}^{-1} = -\frac{1}{4}\left[\,(\partial\phi_{ng})^2 + e^{-2\phi_{ng}}(\partial\chi_{ng})^2\,\right] . \tag{7}$$

By comparing the scalar matrices in (3) and (6), one finds that both matrices are related by the field redefinition

$$e^{\phi_g} = e^{\phi_{ng}} + \chi_{ng}e^{-\phi_{ng}}\chi_{ng} \quad , \quad \chi_g = [\,\chi_{ng} + e^{\phi_{ng}}\chi_{ng}^{-1}e^{\phi_{ng}}\,]^{-1} . \tag{8}$$

The relation between their corresponding representatives in (2) and (5) further requires a field dependent $O \in \mathrm{SO}(2)$ tranformation generated by $E_{\text{comp}} = (E_+ - E_-)$. This is given by

$$\mathcal{V}_g(\phi_g, \chi_g) = O\,\mathcal{V}_{ng}(\phi_{ng}, \chi_{ng}) \quad , \quad O = e^{\xi\,E_{\text{comp}}} \quad \text{with} \quad \xi = \tan^{-1}[e^{-\phi_{ng}}\chi_{ng}] . \tag{9}$$

The physical *unitary* choice

The unitary gauge is recovered by exponentiating the Cartan and the non-compact combination $E_{\text{non-comp}} = (E_+ + E_-)$ orthogonal to E_{comp} in (9). In this physical gauge, the representative takes the form

$$\mathcal{V}_u = e^{\phi_u/2\,H}\, e^{\chi_u\,E_{\text{non-comp}}} = \begin{pmatrix} e^{\phi_u/2}\cosh\chi_u & e^{\phi_u/2}\sinh\chi_u \\ e^{-\phi_u/2}\sinh\chi_u & e^{-\phi_u/2}\cosh\chi_u \end{pmatrix} . \tag{10}$$

The scalar matrix is now given by

$$\mathcal{M}_u = \mathcal{V}_u^t\,\mathcal{V}_u = \begin{pmatrix} \cosh\phi_u\,\cosh 2\chi_u + \sinh\phi_u & \cosh\phi_u\,\sinh 2\chi_u \\ \cosh\phi_u\,\sinh 2\chi_u & e^{-\phi_u}\,\cosh^2\chi_u + e^{\phi_u}\,\sinh^2\chi_u \end{pmatrix} , \tag{11}$$

and the SL(2) invariant non-linear sigma model reads

$$L_u = \frac{1}{8}\,\mathrm{Tr}\,\partial\mathcal{M}_u\partial\mathcal{M}_u^{-1} = -\frac{1}{4}(\partial\phi_u)^2 - \cosh^2\phi_u\,(\partial\chi_u)^2 . \tag{12}$$

By comparing the scalar matrices in (3) and (11), both matrices are this time related by the field redefinition

$$e^{\phi_g} = \cosh 2\chi_u \, \cosh \phi_u + \sinh \phi_u \quad , \quad \chi_g = \frac{\sinh 2\chi_u \, \cosh \phi_u}{\cosh 2\chi_u \, \cosh \phi_u + \sinh \phi_u} \, . \tag{13}$$

The relation between their corresponding representatives in (2) and (10) again requires a field dependent $O \in \mathrm{SO}(2)$ compact tranformation

$$\mathcal{V}_g(\phi_g, \chi_g) = O \, \mathcal{V}_u(\phi_u, \chi_u) \quad , \quad O = e^{\xi \, E_{\mathrm{comp}}} \quad \text{with} \quad \xi = \tan^{-1}[e^{-\phi_u} \tanh \chi_u] \, . \tag{14}$$

The gauge-free *democratic* choice

A last possibility is not to fix the gauge at all and exponentiate all the generators: Cartan, positive and negative roots. In principle, one is not obliged to make any gauge choice. In this gauge-free formalism, one finds

$$\mathcal{V}_d = e^{\phi/2 \, H} \, e^{\chi_- \, E_-} \, e^{\chi_+ \, E_+} = \begin{pmatrix} e^{\phi/2} & e^{\phi/2} \, \chi_+ \\ e^{-\phi/2} \, \chi_- & e^{-\phi/2} + e^{-\phi/2} \, \chi_- \, \chi_+ \end{pmatrix} \, , \tag{15}$$

producing the scalar matrix

$$\mathcal{M}_d = \mathcal{V}_d^t \, \mathcal{V}_d = \begin{pmatrix} e^{\phi} + \chi_- e^{-\phi} \chi_- & (e^{\phi} + \chi_- e^{-\phi} \chi_-) \, \chi_+ + \chi_- \, e^{-\phi} \\ \chi_+ \, (e^{\phi} + \chi_- e^{-\phi} \chi_-) + e^{-\phi} \, \chi_- & \begin{array}{c} e^{-\phi} + \chi_+ (e^{\phi} + \chi_- e^{-\phi} \chi_-) \chi_+ \\ + e^{-\phi} \chi_- \chi_+ + \chi_+ \chi_- e^{-\phi} \end{array} \end{pmatrix} \, , \tag{16}$$

which reduces to the geometric gauge if taking $\phi = \phi_g$, $\chi_+ = \chi_g$ and $\chi_- = 0$ and also to the non-geometric gauge if $\phi = \phi_{ng}$, $\chi_- = \chi_{ng}$ and $\chi_+ = 0$. Building the SL(2) invariant non-linear sigma model from $\mathcal{M}_d$ in (16), one finds non-vanishing kinetic terms for the three fields ϕ, χ_+ and χ_-, so they propagate. This differs from the previous cases where only two fields propagate after fixing the gauge. The extra degree of freedom is not physical in the coset space and can always be removed by applying a field redefinition combined with a compact transformation. By doing so, we can recover any of the previous gauge choices. For instance, redefining the fields as

$$e^{\phi_g} = e^{\phi} + \chi_- e^{-\phi} \chi_- \quad , \quad \chi_g = \chi_+ + [\, \chi_- + e^{\phi} \chi_-^{-1} e^{\phi} \,]^{-1} \, , \tag{17}$$

we go back to the geometric gauge after the local $O \in \mathrm{SO}(2)$ transformation

$$\mathcal{V}_g(\phi_g, \chi_g) = \mathcal{V}_d(\phi_d, \chi_+, \chi_-) \, O \quad , \quad O = e^{\xi \, E_{\mathrm{comp}}} \quad \text{with} \quad \xi = \tan^{-1}[e^{-\phi} \chi_-] \, . \tag{18}$$