FROM STRING TO 4D (AND BACK?)

① From strings to 10D (SUSY) field theories

PARTICLE EVOLUTION IN D DIMENSIONS

$\bullet \Rightarrow X_M(\tau)$ ← proper time

STRING EVOLUTION IN D DIMENSIONS

$\Rightarrow X_M(\tau,\sigma)$, $\quad l_s \sim \sqrt{\alpha'}$

$+\text{ SUSY} \Rightarrow \left.\begin{array}{l}\theta^1(\tau,\sigma)\\ \theta^2(\tau,\sigma)\end{array}\right\}$ Grassman coordinates

SET $D=10$ and $\theta^{1,2}$ being M-W

→ 2D field theory : $X_M(\tau,\sigma)$ and $\theta^{1,2}(\tau,\sigma)$

$$X_M \supset \sum_n a_M^{(n)} e^{-2in(\tau-\sigma)} + \sum_n \tilde{a}_M^{(n)} e^{-2in(\tau+\sigma)}$$

τ, σ: X_M, θ^1, θ^2

$$\Rightarrow \theta^1 \supset \sum_n b^{(n)} e^{-2in(\tau-\sigma)} \ ; \ \theta^2 \supset \sum_n \tilde{b}^{(n)} e^{-2in(\tau+\sigma)}$$

Promote a's, $\tilde{a}$'s, b's, $\tilde{b}$'s to operators with $[\,,\,]$ or $\{\,,\,\}$ relations:

$|\text{state}\rangle = a_M^\dagger a_N^\dagger |0\rangle = \underset{\text{10D: metric}}{g_{MN}} \oplus \underset{\text{antisym}}{B_{MN}} \oplus \underset{\text{scalar}}{\Phi}$

→ Mass of a state :

$$M^2 = \frac{1}{l_s^2}\Big[\underbrace{N(a,b) + \tilde{N}(\tilde{a},\tilde{b})}_{\text{occupation number}}\Big] \quad \xrightarrow{l_s \to 0} \quad \begin{array}{l} M^2 \to \infty \\ 0 \to 0 \end{array}$$

⇒ keep only massless states!!

→ 10D massless spectrum: $\overbrace{g_{MN},\ B_{(2)},\ \Phi,\ C_{(p)}}^{\text{BOSONS}}\ ;\ \overbrace{\chi_{1,2}\ ;\ \Psi_{1,2}}^{\text{FERMIONS}}$

$p=1,3 \to$ IIA $[ch\,\theta^1 \neq ch\,\theta^2]$
$p=0,2,4 \to$ IIB $[ch\,\theta^1 = ch\,\theta^2]$

• Lagrangian :

$$\boxed{\mathcal{L}_{10D} = \sqrt{g}\left[R[g] - \frac{1}{2}\partial_M\Phi\,\partial^M\Phi - \frac{1}{12}e^{-\Phi} H_{MNP}H^{MNP} + \ldots + \text{fermi}\right]}$$

$H_{(3)} \equiv H_{MNP} = \partial_M B_{NP} + \text{permut}$

• One can study a test string propagating in a background $\{g_{MN}, B_{MN}, \phi, C_{(p)}\}$ generated by the strings around

$$S_{\substack{test\\string}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \,(\partial_\alpha X^M)(\partial^\alpha X^N)\underbrace{g_{MN}(X)} + \epsilon^{\alpha\beta}(\partial_\alpha X^M)(\partial_\beta X^N)\underbrace{B_{MN}(x)} + \dots$$

$\sigma^\alpha = (\tau, \sigma)$

The fields are the couplings of the 2d field theory:

$$\boxed{\text{Conformal invariance Request!!} \Rightarrow \underbrace{\beta_g^{MN} = \beta_B^{MN} = \dots\dots = 0}_{\text{At lowest level in } \frac{\sqrt{\alpha'}}{L} \Rightarrow \text{E.O.M of an action!!}}}$$

L — scale of system

$$S_{\substack{SUGRA\\10D}} = \frac{1}{2K^2}\int d^{10}x\,\sqrt{-g}\left(R - \frac{1}{2}\partial^M\phi\,\partial_M\phi - \frac{1}{12}e^{-\phi}\underbrace{H_{MNP}\,H^{MNP}}_{H_{(3)} = dB_{(2)} = \partial_M B_{NP} + \text{permut.}}\right)$$

$$-\frac{1}{4K^2}\int dx^{10}\sqrt{-g}\begin{cases}\frac{1}{2}e^{\frac{3}{2}\phi}\hat{F}_{MN}\hat{F}^{MN} + \frac{1}{24}e^{\frac{\phi}{2}}\hat{F}_{M_1\dots M_4}\hat{F}^{M_1\dots M_4} & \text{(Type IIA)}\\ e^{2\phi}\partial^M C_0\,\partial_M C_0 + \frac{1}{6}e^{\phi}\hat{F}_{MNP}\hat{F}^{MNP} + \frac{1}{240}\hat{F}_{M_1\dots M_5}\hat{F}^{M_1\dots M_5} & \text{(Type IIB)}\end{cases}$$

$$+ \underbrace{-\frac{1}{4K^2}\int\begin{cases}B_{(2)}\wedge F_{(4)}\wedge F_{(4)}\\ C_{(4)}\wedge H_{(3)}\wedge F_{(3)}\end{cases}}_{\text{C-S terms (topological)}} + S_{\text{Fermionic}}(\chi_{1,2}, \psi_{1,2})$$

where the gauge-invariant field strengths are given by:

IIA: $\hat{F}_{(2)} = F_{(2)} = d\,C_{(1)}$

$\hat{F}_{(4)} = \underbrace{F_{(4)}}_{dC_{(3)}} + C_{(1)}\wedge H_{(3)}$

IIB: $\hat{F}_{(3)} = \underbrace{F_{(3)}}_{dC_{(2)}} - H_{(3)}\wedge C_0$

$\hat{F}_{(5)} = \underbrace{F_{(5)}}_{dC_{(4)}} + \frac{1}{2}(B_{(2)}\wedge F_{(3)} - C_{(2)}\wedge H_{(3)})$

⇒ We have obtained a susy $\underbrace{\text{extension}}_{\{B_{(2)}, \phi, C_{(p)}\}}$ of gravity in 10D !!

③ FROM 10D SUGRA TO 4D SUGRA : K-K REDUCTION

- K·K reduction : Massless Scalar field in $5D = \mathbb{R}_{1,3} \times S^1$

$$\hat{\phi}(x^\mu, y) = \sum_{-\infty}^{\infty} \hat{\phi}^{(n)}(x)\, e^{i\frac{n\cdot y}{L}}$$ → Radius of S^1

* Massless in 5D ⟹ massive in 4D

$$\hat{\Box}\,\hat{\phi}(x,y) = 0 \implies \Box\,\hat{\phi}^{(n)}(x) - \underbrace{\left(\frac{n}{L}\right)^2}_{m^2}\hat{\phi}^{(n)}(x) = 0$$

* Truncation to $\hat{\phi}^{(0)}(x)$:

i) Enough for dynamics @ $E \ll \frac{1}{L}$ [Physics]

ii) Consistent with the higher-dim. E.O.M : $\Box\,\hat{\phi}^{(n\neq 0)}(x) \neq \hat{\phi}^{(0)}(x)$ "never sourced" [Group-Theory]

- K·K reduction : Massless Pure gravity in $5D = \mathbb{R}_{1,3} \times S^1$.

$$\hat{g}_{MN}(x^\mu, y) = \sum_{-\infty}^{\infty} \hat{g}^{(n)}_{MN}(x)\cdot e^{i\frac{n\cdot y}{L}}$$

* Truncation to $\hat{g}^{(0)}_{MN}(x)$

* Dimensional reduction :

$$\hat{g}^{(0)}_{MN}(x) = \begin{bmatrix} \hat{g}^{(0)}_{\mu\nu} & \hat{g}^{(0)}_{\mu y} \\ \hat{g}^{(0)}_{y\mu} & \hat{g}^{(0)}_{yy} \end{bmatrix}$$

metric in 4D ; vector in 4D ; scalar in 4D

* Gravity in 5D ⟹ Einstein - Maxwell - dilaton theory in 4D

$$\mathcal{L}_{5D} = -\sqrt{\hat{g}}\;\hat{R}[\hat{g}] \;\xrightarrow[S^1]{}\; \mathcal{L}_{4D} = \sqrt{-g}\left[R[g] - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}\,\underbrace{|F|^2}_{F_{\mu\nu}F^{\mu\nu}}\right]$$

where $\hat{g}^{(0)}_{\mu\nu} = e^{\frac{1}{\sqrt{3}}\phi}\, g_{\mu\nu} + e^{-\frac{2}{\sqrt{3}}\phi} A_\mu A_\nu$, $\hat{g}^{(0)}_{y\mu} = e^{-\frac{2}{\sqrt{3}}\phi} A_\mu$, $\hat{g}^{(0)}_{yy} = e^{-\frac{2}{\sqrt{3}}\phi}$

i) $$\underbrace{GL(5)}_{\xi^M(x,y)} \longrightarrow \underbrace{\underbrace{GL(4)}_{\xi^\mu(x)} \times U(1)_{gauge} \times \boxed{\mathbb{R} \quad \begin{matrix}\phi \to \phi + c \\ A_\mu \to e^{c\frac{\sqrt{3}}{2}} A_\mu\end{matrix}}}_{\xi^y(x,y) = \Lambda(x) + c\cdot y}$$

→ Enhancement with a global symmetry !! $G_{global} = \mathbb{R}$

ii) ϕ is uncharged under the $U(1)_{gauge}$ ⟹ the charge of $\hat{g}^{(n)}_{yy} = \left(\frac{n}{L}\right)$

Massless

- KK-reduction : Pure gravity in 10D.
  - Perform a chain of S^1 reductions $S^1_{(1)} \otimes S^1_{(2)} \otimes \cdots \otimes S^1_{(6)} = T^6$.

$$\underbrace{g_{MN}}_{GL(10)} \xrightarrow{S^1_{(1)}} \underbrace{\begin{matrix} g_{\mu\nu} \\ A_\mu^{(1)} \\ \phi^{(1)} \end{matrix}}_{\substack{GL(9)\times U(1)_{gauge}\times G_{glob} \\ \text{with } G_{glob}=\mathbb{R}}} \xrightarrow{S^1_{(2)}} \underbrace{\begin{matrix} g_{\mu\nu},\ A_\mu^{(2)},\ \phi^{(2)} \\ A_\mu^{(1)},\ \phi^{(1,2)} \\ \phi^{(1)} \end{matrix}}_{\substack{GL(8)\times U(1)^2_{gauge}\times G_{glob} \\ \text{with } G_{glob}=\mathbb{R}\times SL(2)}} \xrightarrow{S^1_{(3)}} \cdots$$

$$\cdots \xrightarrow{S^1_{(6)}} \underbrace{\begin{matrix} g_{\mu\nu} & (1 \text{ metric}) \\ A_\mu^{(1)},\dots,A_\mu^{(6)} & (6 \text{ vectors}) \\ \phi^{(1)},\dots,\phi^{(6)} & (6 \text{ dilatons}) \\ \phi^{(i,j)}\ \ i<j & (15 \text{ axions}) \end{matrix}}_{\substack{GL(4)\times U(1)^6_{gauge}\times G_{glob} \\ \text{with } G_{glob}=\mathbb{R}\times SL(6)}} \Bigg\} \text{ 21 scalars uncharged under } U(1)$$

- K-K reduction : Type IIA / IIB supergravity in 10D

$$\underbrace{\begin{matrix} \text{univ}: g_{MN},\ B_{MN},\ \Phi \\ \text{non univ}: C_{(p)}\ [IIA/IIB] \\ \text{fermi}\begin{cases} \chi_1,\chi_2 \\ \Psi_1,\Psi_2 \end{cases} \end{matrix}}_{\substack{GL(10) \\ \mathcal{N}=2 \text{ SUSY in 10D}}} \xrightarrow{S^1_{(1)}} \cdots \xrightarrow{S^1_{(6)}} \underbrace{\underbrace{\begin{matrix} g_{\mu\nu} & (1 \text{ metric}) \\ A_\mu^{(1)},\dots,A_\mu^{(28)} & (28 \text{ vectors}) \\ \phi^{(1)},\dots,\phi^{(70)} & (70 \text{ scalars}) \to \text{uncharged} \\ \chi_{(1)},\dots,\chi_{(56)} & (56 \text{ dilatini} - 1/2) \\ \Psi_{(1)},\dots,\Psi_{(8)} & (8 \text{ gravitini} - 3/2) \end{matrix}}_{\left[GL(4)\times U(1)^{28}_{gauge}\times G_{glob}\right] \times \underbrace{SU(8)}_{R\text{-Symmetry}} \ \text{with } G_{glob}=E_{7(7)}}}_{\mathcal{N}=8 \text{ SUPERGRAVITY IN 4D !!}}$$

PROBLEM

$\Rightarrow$ Toroidal reductions produce abelian [ungauged] supergravities in 4D, with uncharged scalars [moduli] which are also free, i.e. $V(\phi)=0$.

$\hookrightarrow$ How to get moduli stabilisation ??

(3) FROM 10D SUGRA TO 4D SUGRA : BEYOND T^6 REDUCTION

- How to get interacting (non-abelian, $V(\phi) \neq 0$) 4D theories ?

  i) Different internal geometries and dependences of the fields [SS]

$\mathcal{M}_6 \neq T^6$
  1) Twisted tori : $\underbrace{de^m = 0}_{\text{untwisted}} \longrightarrow \underbrace{de^m = \frac{1}{2}\, \omega^m{}_{np}\, e^n \wedge e^p}_{e^m = U^m{}_n(y)\, dy^n}$, $\omega^m{}_{np}$: twist $\equiv$ struc. const. of the twist group
  2) 6-dim group manifolds G : $G_{isom} = G_L \times G_R$
  3) 6-dim coset spaces G/H : $G_{isom} = G$ $[S^n, CP^n]$

  ii) Turning on fluxes, i.e. $H_{(3)} = dB_{(2)} + \underbrace{H^{flux}_{(3)}}_{cte}$, $F_{(p+1)} = dC_{(p)} + \underbrace{F^{flux}_{(p+1)}}_{cte}$ along internal space directions y^m.

$\Rightarrow$ Non-abelian and non-free 4D theory !!

* non-abelian symmetry : $f_{ij}{}^k = f_{ij}{}^k(\underbrace{\omega^m{}_{np}}_{\substack{\text{twist}\\ G_{isom}}}, \underbrace{H^{flux}_{(3)}, F^{flux}_{(p+1)}}_{\text{fluxes}}) \equiv$ structure constant of the non-abelian gauge symmetry spanned by the 28 vectors $A_\mu^{i=1,\dots,28}$

$$\boxed{\begin{array}{c}[T_i, T_j] = f_{ij}{}^k\, T_k \\ \text{28-dim gauge algebra}\end{array}}$$

$i,j,k = 1,\dots,28$

* scalar potential (charged scalars) : $V(\underbrace{\vec{\phi}}_{\substack{70\\ \text{fields}}} ; \underbrace{\omega^m{}_{np}, H^{flux}_{(3)}, F^{flux}_{(p+1)}}_{\text{couplings}}) \equiv$ moduli stabilisation
Critical points and $V(\vec{\phi}_0) \equiv$ c.c
No stable dS solution

- With the ingredients $\{\omega^m{}_{np}, H^{flux}_{(3)}, F^{flux}_{(p+1)}\}$, can we obtain the most general 4D supergravity preserving $\mathcal{N}=8$ with a general non-abelian gauge symmetry ? $\boxed{\text{NO !!}}$

- Is there a framework to describe any 4D supergravity with $\mathcal{N}=8$ susy with any 28-dim non-abelian gauge group ?

$\boxed{\text{YES !!}}$

④ THE MOST GENERAL $\mathcal{N}=8$, $D=4$ GAUGED SUPERGRAVITY

- In the ungauged case $[T^6 + \text{no-fluxes}] \Rightarrow \underbrace{G_{global} = E_{7(7)}}_{dim=133}$; $\underbrace{G_{gauge} = U(1)^{28}}_{dim=28}$

- The most general and consistent $\mathcal{N}=8$, $D=4$ supergravity can be obtained by promoting a subgroup of $G_{global} = E_{7(7)}$ to be the non-abelian $G_{gauge} \Rightarrow G_{gauge} \subset E_{7(7)}$.

$$\underbrace{T_i}_{\substack{28\text{-generators}\\ \text{of } G_{gauge}}} = \Theta_i{}^{\alpha} \underbrace{\tau_\alpha}_{\substack{133\text{-generators}\\ \text{of } E_{7(7)}}} \longrightarrow \text{Embedding tensor !! (Selector)}$$

→ gauge algebra : $[T_i, T_j] = f_{ij}{}^{k}\, T_k$ with $f_{ij}{}^{k}(\Theta) = $ struc. const.

→ covariant derivatives : $D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + \Theta\, A_\mu \vec{\phi}$ ↳ charges

→ scalar potential : $V(\vec{\phi}, \underbrace{\Theta}_{\text{couplings}})$

→ $\Theta_i{}^{\alpha} \in \mathbf{912}$ of $E_{7(7)} \Rightarrow 912$ independent components !!

- Finger counting : $\underbrace{90}_{\omega_{mn}^{p}} + \underbrace{20}_{H_{(3)}^{flux}} + \underbrace{32}_{F_{(p+1)}^{flux}} = 142$ components.

$$\boxed{142 < 912} \Rightarrow \text{Non-geometric fluxes !!}$$

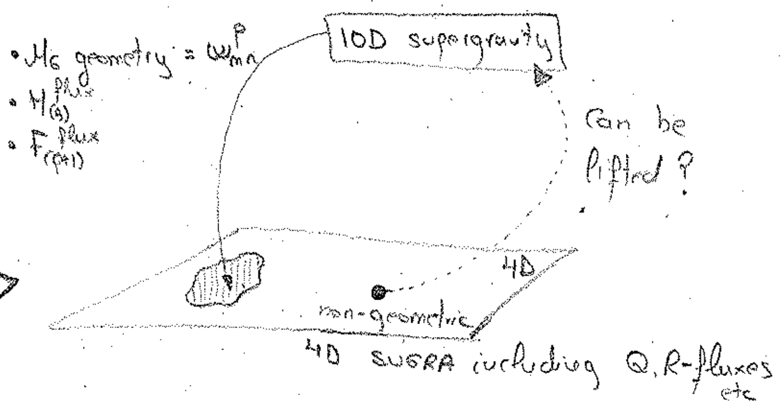
(pure 4D approach to match $\Theta_i{}^{\alpha}$)

Q_{p}^{mn}, R^{mnp}, ...

Q-flux ← → R-flux

- Non-geometric fluxes can produce stable dS critical points !!

- Higher-dimensional origin ? ⟹



⑤ HIGHER – DIMENSIONAL ORIGIN OF NON-GEOMETRIC FLUXES

→ Why should it exist? ⟹ Belief in string theory ⟹ Reality is trully 10D

→ Non-geometric 4D theories seem to be related to trully "stringy" configurations : Q-fluxes ⟺ winding modes of strings

$\Downarrow$

$\boxed{l_s \sim L}$ ⟹ "breakdown" $\xrightarrow{\alpha' \to 0}$.

$\Downarrow$

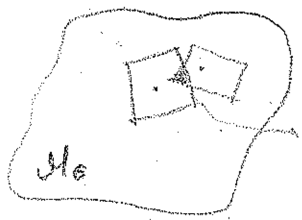
$\boxed{\text{Reconsider the limits } l_s \ll L \ll \text{SYSTEM}}$

→ How do strings perceive geometry ?

i) (Exceptional) Generalised geometry

ii) Doubled geometry / DFT

iii) Non-commutative / Non associative geometry ?



$GL(6) \to E_{7(7)}$ transition functions.

→ Is there a fundamental meaning of lower – dimensional emergent symmetries or it is just an "artifact" of the dimensional reduction prescription ?